

# UNIQUENESS OF TOPOLOGICAL MULTI-VORTEX SOLUTIONS FOR A SKEW-SYMMETRIC CHERN-SIMONS SYSTEM

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ABSTRACT. Consider the following skew-symmetric Chern-Simons system

$$\begin{cases} \Delta u_1 + \frac{1}{\varepsilon^2} e^{u_2} (1 - e^{u_1}) = 4\pi \sum_{j=1}^{N_1} \delta_{p_{j,1}} \\ \Delta u_2 + \frac{1}{\varepsilon^2} e^{u_1} (1 - e^{u_2}) = 4\pi \sum_{j=1}^{N_2} \delta_{p_{j,2}} \end{cases} \quad \text{in } \Omega,$$

where  $\Omega$  is a flat 2-dimensional torus  $\mathbb{T}^2$  or  $\mathbb{R}^2$ ,  $\varepsilon > 0$  is a coupling parameter, and  $\delta_p$  denotes the Dirac measure concentrated at  $p$ . In this paper, we prove that, when the coupling parameter  $\varepsilon$  is small, the topological type solutions to the above system are uniquely determined by the location of their vortex points. This result follows by the bubbling analysis and the non-degeneracy of linearized equations.

## 1. INTRODUCTION

In recent years, various Chern–Simons models have been proposed to study condensed matter physics and particle physics, such as the relativistic Chern-Simons models of high temperature superconductivity [16, 9], Lozano-Marqués-Moreno-Schaposnik model [21] of bosonic sector of  $\mathcal{N} = 2$  supersymmetric Chern-Simons-Higgs theory, and Gudnason model [12, 13] of  $\mathcal{N} = 2$  supersymmetric Yang-Mills-Chern-Simons-Higgs theory and so on. The relative Euler–Lagrange equations of those models often provided many mathematical challenging problems. We refer the readers to [9, 31] for exhaustive bibliography.

Speilman et al.[23] observed no parity breaking in the experiment with high temperature superconductivity. Hagen[14] and Wilczek[27] indicated the parity broken may not happen in the a field theory with even number of Chern-Simons gauge fields. One of the simplest models of this kind is the  $[U(1)]^2$  Chern-Simons model of two Higgs fields, where each of them coupled to one of two Chern–Simons fields. In this paper, we will

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study the relativistic self-dual  $[U(1)]^2$  Chern-Simons model proposed by Kim et al[18]. For simplicity, we consider the case with only mutual Chern-Simons interaction. We give only a brief description on this model. Let  $(A_\mu^{(i)})$  ( $\mu = 0, 1, 2, i = 1, 2$ ) be two Abelian gauge fields and  $\phi_i$  ( $i = 1, 2$ ) be two Higgs scalar fields, where the electromagnetic fields and covariant derivatives are defined by

$$F_{\mu\nu}^{(i)} = \partial_\mu A_\nu^{(i)} - \partial_\nu A_\mu^{(i)}, \quad D_\mu \phi_i = \partial_\mu \phi_i - i A_\mu^{(i)} \phi_i, \quad \mu = 0, 1, 2, \quad i = 1, 2. \quad (1.1)$$

The Lagrangian of this model is written in the form

$$\mathcal{L} = -\frac{\varepsilon}{2} \epsilon^{\mu\nu\alpha} (A_\mu^{(1)} F_{\nu\alpha}^{(2)} + A_\mu^{(2)} F_{\nu\alpha}^{(1)}) + \sum_{i=1}^2 D_\mu \phi_i \overline{D^\mu \phi_i} - V(\phi_1, \phi_2), \quad (1.2)$$

where  $\varepsilon > 0$  is a coupling parameter, and the Higgs potential  $V(\phi_1, \phi_2)$  is taken as

$$V(\phi_1, \phi_2) = \frac{1}{4\varepsilon^2} \left( |\phi_2|^2 [|\phi_1|^2 - 1]^2 + |\phi_1|^2 [|\phi_2|^2 - 1]^2 \right). \quad (1.3)$$

After a BPS reduction [1, 22], one can show that the energy minimizer satisfies the following self-dual equation:

$$\begin{cases} D_1 \phi_k \pm i D_2 \phi_k = 0, & k = 1, 2, \\ F_{12}^{(1)} \pm \frac{1}{2\varepsilon^2} |\phi_2|^2 (|\phi_1|^2 - 1) = 0, \\ F_{12}^{(2)} \pm \frac{1}{2\varepsilon^2} |\phi_1|^2 (|\phi_2|^2 - 1) = 0. \end{cases} \quad (1.4)$$

As in [17], we let  $u_{\varepsilon,i} = \ln |\phi_i|^2$ , and denote the zeros of  $\phi_i$  by  $\{p_{1,i}, \dots, p_{N_i,i}\}$ ,  $i = 1, 2$ . Then  $(u_{\varepsilon,1}, u_{\varepsilon,2})$  satisfies

$$\begin{cases} \Delta u_{\varepsilon,1} + \frac{1}{\varepsilon^2} e^{u_{\varepsilon,2}} (1 - e^{u_{\varepsilon,1}}) = 4\pi \sum_{j=1}^{N_1} \delta_{p_{j,1}} & \text{on } \Omega, \\ \Delta u_{\varepsilon,2} + \frac{1}{\varepsilon^2} e^{u_{\varepsilon,1}} (1 - e^{u_{\varepsilon,2}}) = 4\pi \sum_{j=1}^{N_2} \delta_{p_{j,2}} & \text{on } \Omega, \end{cases} \quad (1.5)$$

where  $\delta_p$  is the Dirac measure at  $p$ . See [18, 10, 19] for the details of the derivation of (1.5) from (1.4).  $\Omega$  here is usually referred to  $\mathbb{R}^2$  or a flat torus  $\mathbb{T}^2$ .

When  $u_{\varepsilon,1} = u_{\varepsilon,2} = u_\varepsilon$  and  $\sum_{j=1}^{N_1} \delta_{p_{j,1}} = \sum_{j=1}^{N_2} \delta_{p_{j,2}} = \sum_{j=1}^N \delta_{p_j}$ , then (1.5) is reduced to

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2} e^{u_\varepsilon} (1 - e^{u_\varepsilon}) = 4\pi \sum_{j=1}^N \delta_{p_j}, \quad (1.6)$$

which is the equation derived from the Abelian Chern-Simons model with one Higgs particles. See [16] for the physical background. Compared to (1.5), the equation (1.6) has

been studied extensively in the last two decades. We refer [3, 4, 6, 7, 8, 9, 24, 25, 26] and reference therein for more details.

On the other hand, the system (1.5) is a typical *skew-symmetric system*. We introduce the background functions on  $\mathbb{T}^2$  to remove the singularities.

$$u_{0,i} = -4\pi \sum_{j=1}^{N_i} G(x, p_{j,i}), \quad i = 1, 2, \quad (1.7)$$

where  $G(x, q)$  is the Green function defined by

$$\begin{cases} -\Delta G(x, q) = \delta_q - \frac{1}{|\mathbb{T}^2|}, \\ \int_{\mathbb{T}^2} G(x, q) dx = 0, \end{cases} \quad (1.8)$$

and  $|\mathbb{T}^2|$  is the area of  $\mathbb{T}^2$ . With the transform  $u_{\varepsilon,i} \rightarrow u_{0,i} + u_{\varepsilon,i}$ ,  $i = 1, 2$ , the system (1.5) can be reduced into

$$\begin{cases} \Delta u_{\varepsilon,1} + \frac{1}{\varepsilon^2} e^{u_{0,2}+u_{\varepsilon,2}} (1 - e^{u_{0,1}+u_{\varepsilon,1}}) = \frac{4N_1\pi}{|\mathbb{T}^2|} \\ \Delta u_{\varepsilon,2} + \frac{1}{\varepsilon^2} e^{u_{0,1}+u_{\varepsilon,1}} (1 - e^{u_{0,2}+u_{\varepsilon,2}}) = \frac{4N_2\pi}{|\mathbb{T}^2|} \end{cases} \quad \text{in } \mathbb{T}^2. \quad (1.9)$$

Then any solution of the system (1.9) is a critical point of the following functional

$$\begin{aligned} I(u_{\varepsilon,1}, u_{\varepsilon,2}) &= \int_{\mathbb{T}^2} \left\{ \frac{1}{2} \nabla u_{\varepsilon,1} \cdot \nabla u_{\varepsilon,2} + \frac{1}{\varepsilon^2} (1 - e^{u_{0,1}+u_{\varepsilon,1}}) (1 - e^{u_{0,2}+u_{\varepsilon,2}}) \right. \\ &\quad \left. + \frac{4\pi}{|\mathbb{T}^2|} (N_1 u_{\varepsilon,1} + N_2 u_{\varepsilon,2}) \right\} dx \end{aligned} \quad (1.10)$$

We refer the readers to [29, 30, 28] for more information about skew-symmetric systems. Since the action functional (1.10) is indefinite, there are difficulties of studying (1.5) from the direct variational method.

From the potential energy density, it can be seen that the finite energy condition impose the following behaviors of  $(u_{\varepsilon,1}, u_{\varepsilon,2})$ :

**a.**  $\Omega = \mathbb{R}^2$

$$(1) \quad (u_{\varepsilon,1}, u_{\varepsilon,2}) \rightarrow (0, 0) \quad \text{as } |x| \rightarrow \infty.$$

$$(2) \quad (u_{\varepsilon,1}, u_{\varepsilon,2}) \rightarrow (-\infty, -\infty) \quad \text{as } |x| \rightarrow \infty.$$

**b.**  $\Omega = \text{flat torus } \mathbb{T}^2$

$$(1) \quad (u_{\varepsilon,1}, u_{\varepsilon,2}) \rightarrow (0, 0) \quad \text{a.e. as } \varepsilon \rightarrow 0.$$

$$(2) \quad (u_{\varepsilon,1}, u_{\varepsilon,2}) \rightarrow (-\infty, -\infty) \quad \text{a.e. as } \varepsilon \rightarrow 0.$$

In the physical literature, a solution to (1.5) satisfies **a**.(1) or **b**.(1) is called a *topological solution* and satisfies **a**.(2) or **b**.(2) is called a *non-topological solution*.

Lin, Ponce and Yang [19] initiated the mathematical study on this system, where they established the existence of topological solution in  $\mathbb{R}^2$ . Since the main difficulty arises from the skew-symmetric structure of the system (1.5), they used the constrained minimization method with a deep application of Moser-Trudinger inequality. Since then, this system (1.5) has been studied from other aspects, such as the existence of topological solutions over a flat torus [20], the existence of non-topological solutions over the plane and a flat torus [15] and the structure of the radial solutions over the plane [5].

In [20], Lin and Prajapat applied a monotone scheme and the constrained minimization method to obtain two kind of solutions to (1.5) over a flat torus: maximal solution and mountain-pass solution. Here,  $(u_{\varepsilon,1}, u_{\varepsilon,2})$  is called a maximal solution to (1.5) if

$$u_i < u_{\varepsilon,i}, \quad i = 1, 2,$$

for other solutions  $(u_1, u_2)$  to (1.5). Furthermore, they showed that the maximal solution is unique when  $\varepsilon > 0$  is small. It is obvious that the maximal solution is a topological solution. Naturally, we are lead to the question *whether the topological solution is unique*. We give a positive answer to this question when  $\varepsilon > 0$  is small.

**Theorem 1.1.** *Consider  $\Omega = \mathbb{T}^2$ . There exists  $\varepsilon_0 := \varepsilon_0(p_{j,i}) > 0$  such that there exists a unique topological solution of (1.5) for each  $\varepsilon \in (0, \varepsilon_0)$ . Moreover, any topological solution is a unique maximal solution of (1.5) for  $\varepsilon \in (0, \varepsilon_0)$ .*

It is worth to note that when  $u_{\varepsilon,1} = u_{\varepsilon,2} = u_\varepsilon$  and  $\sum_{j=1}^{N_1} \delta_{p_{j,1}} = \sum_{j=1}^{N_2} \delta_{p_{j,2}} = \sum_{j=1}^N \delta_{p_j}$ , our theorem is reduced to the uniqueness theorem for the topological solutions to the scalar equation (1.6) on  $\mathbb{T}^2$  proved by Choe [6] (also on  $\mathbb{R}^2$ ) and Tarantello [26] independently. Choe [6] showed that the topological solution can be approximated by the sum of rescaled radial topological solution and used the invertibility of the linearized operator from  $W^{2,2}$  to  $L^2$  to prove the uniqueness of the topological solutions. Tarantello [26] showed that the topological solutions to (1.6) is a strict local minimum for the corresponding action functional and the uniqueness follows. On the other hand, since our problem has indefinite functional, it is difficult to use the concept of stability (local minimizer). So we use different approach such that we observe the behavior of the direct

difference of two topological solutions with  $L^\infty$ -normalization instead of  $W^{2,2}$  or  $L^2$ . In our proof of Theorem 1.1, to prove the uniqueness of topological solution for small  $\varepsilon$  on  $\mathbb{T}^2$ , we investigate the behavior of topological solutions as  $\varepsilon \rightarrow 0$  for (1.5) in Section 2 as a generalization of the estimates obtained in [6, 26]. In fact, the similar arguments on  $\mathbb{T}^2$  in Section 2 also hold for the topological entire solutions on  $\mathbb{R}^2$  due to the fact that the topological entire solutions achieve the boundary condition exponentially fast at infinity. More precisely, Lin Ponce and Yang proved the following theorem.

**Theorem A.** [19] *Suppose  $(u_{\varepsilon,1}, u_{\varepsilon,2})$  is a topological solution of (1.5) in  $\mathbb{R}^2$ . Then*

$$\sum_{i=1}^2 (|u_{\varepsilon,i}(x)| + |\nabla u_{\varepsilon,i}(x)|) \leq C \frac{e^{-\frac{|x|}{\varepsilon}}}{|x|^{1/2}} \quad (1.11)$$

for some constant  $C$  and  $|x|$  sufficiently large.

In view of the above good exponential decay property of the topological entire solutions, we obtain the following theorem.

**Theorem 1.2.** *Consider  $\Omega = \mathbb{R}^2$ . There exists  $\varepsilon_0 := \varepsilon_0(p_{j,i}) > 0$  such that there exists a unique topological entire solution of (1.5) for each  $\varepsilon \in (0, \varepsilon_0)$ .*

Firstly, we sketch our proof for Theorem 1.1 here. Suppose, for the sake of contradiction, that there exist two sequences of distinct topological solutions  $(u_{\varepsilon,1}, u_{\varepsilon,2})$  and  $(\tilde{u}_{\varepsilon,1}, \tilde{u}_{\varepsilon,2})$  of (1.5). Without loss of generality, we may assume that there exists  $x_\varepsilon \in \mathbb{T}^2$  such that

$$|u_{\varepsilon,1}(x_\varepsilon) - \tilde{u}_{\varepsilon,1}(x_\varepsilon)| = \|u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)} \geq \|u_{\varepsilon,2} - \tilde{u}_{\varepsilon,2}\|_{L^\infty(\mathbb{T}^2)}$$

and  $x_\varepsilon \rightarrow p$  as  $\varepsilon \rightarrow 0$  (up to subsequence). Set

$$A_\varepsilon \equiv \frac{u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}}{\|u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)}} \quad \text{and} \quad B_\varepsilon \equiv \frac{u_{\varepsilon,2} - \tilde{u}_{\varepsilon,2}}{\|u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)}}. \quad (1.12)$$

Then  $(A_\varepsilon, B_\varepsilon)$  satisfies

$$\begin{cases} \Delta A_\varepsilon - \frac{1}{\varepsilon^2} e^{\tilde{u}_{\varepsilon,2} + \eta_{\varepsilon,1}} A_\varepsilon + \frac{1}{\varepsilon^2} e^{\eta_{\varepsilon,2}} (1 - e^{u_{\varepsilon,1}}) B_\varepsilon = 0 & \text{on } \mathbb{T}^2, \\ \Delta B_\varepsilon - \frac{1}{\varepsilon^2} e^{\tilde{u}_{\varepsilon,1} + \eta_{\varepsilon,2}} B_\varepsilon + \frac{1}{\varepsilon^2} e^{\eta_{\varepsilon,1}} (1 - e^{u_{\varepsilon,2}}) A_\varepsilon = 0 & \text{on } \mathbb{T}^2, \end{cases} \quad (1.13)$$

where  $\eta_{\varepsilon,i}$  is between  $u_{\varepsilon,i}$  and  $\tilde{u}_{\varepsilon,i}$ ,  $i = 1, 2$ . After suitable rescaling at the maximum points (see Section 3), (1.13) converges to a bounded solution  $(A, B)$  of

$$\begin{cases} \Delta A - e^{U_1+U_2}A + e^{U_2}(1 - e^{U_1})B = 0 \\ \Delta B - e^{U_1+U_2}B + e^{U_1}(1 - e^{U_2})A = 0 \end{cases} \quad \text{in } \mathbb{R}^2. \quad (1.14)$$

where  $(U_1, U_2)$  is a topological solution to

$$\begin{cases} \Delta u_1 + e^{u_2}(1 - e^{u_1}) = 4\pi\nu_1\delta_0 \\ \Delta u_2 + e^{u_1}(1 - e^{u_2}) = 4\pi\nu_2\delta_0 \end{cases} \quad \text{in } \mathbb{R}^2. \quad (1.15)$$

and  $\nu_1$  and  $\nu_2$  are constants which are determined by the choice of the rescaling region. By the standard method of moving plane[2], one can show that the topological solution  $(u_1, u_2)$  of (1.15) is radially symmetric with respect to the origin. For any topological solution of (1.15), Chern, Chen and Lin[5] showed the non-degeneracy of the linearized system (1.14), i.e.,  $A = B = 0$ .

**Theorem B.** [5] *Let  $(U_1, U_2)$  be the radial topological solution of (1.15). Then the linearized equation (1.14) of (1.15) at  $(U_1, U_2)$  is non-degenerate, i.e., if  $(A, B)$  is a pair of bounded solution of (1.14), then*

$$(A, B) \equiv (0, 0).$$

*Moreover, equation (1.15) possesses one and only one topological solution.*

Then the uniqueness of the topological solutions of (1.5) on  $\mathbb{T}^2$  follows from Theorem B. Obviously, the main ingredient of our approach is to show how  $(A_\varepsilon, B_\varepsilon)$  would converge to a bounded solution of (1.14).

The main different part between the proof of Theorem 1.1 and Theorem 1.2 is that on  $\mathbb{R}^2$ , the maximum point  $x_\varepsilon$  of  $|u_{\varepsilon,i} - \tilde{u}_{\varepsilon,i}|$  can diverge to  $\infty$  unlike on  $\mathbb{T}^2$ . Even in this case, we can use the good convergence property of topological entire solutions to prove Theorem 1.2 (see the end of Section 3).

This paper is organized as follows. In Section 2, we establish some preliminary estimates for the topological solutions which are important to show that (1.13) converges to (1.14). Section 3 is devoted to the proof of Theorem 1.1-1.2.

## 2. PRELIMINARIES

In this section, we will show some preliminary estimates for the topological solutions to (1.5) in  $\mathbb{T}^2$ . Then in view of Theorem A, the similar arguments in this section are also true for the topological entire solutions on  $\mathbb{R}^2$ . Our main goal in this section is to show the topological solutions to (1.5), after suitable rescaling, can converge to the radially symmetric entire topological solutions on a certain domain (see Lemma 2.4 below).

By maximum principle, it is clear that satisfies  $u_{\varepsilon,i} < 0$  on  $\mathbb{T}^2$  for  $i = 1, 2$ . Thus, by integrating (1.5), we have

$$\int_{\mathbb{T}^2} \frac{1}{\varepsilon^2} e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) dx = \int_{\mathbb{T}^2} \frac{1}{\varepsilon^2} |e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}})| dx = 4\pi N_i, \quad 1 \leq j \neq i \leq 2. \quad (2.1)$$

We show that, there are only two types of solutions, topological and non-topological solutions, as  $\varepsilon \rightarrow 0$ . In particular, if  $(u_{\varepsilon,1}, u_{\varepsilon,2})$  is a topological solution, then  $u_{\varepsilon,i} \rightarrow 0$  ( $i = 1, 2$ ) in  $L^p(\mathbb{T}^2)$  for some  $p > 1$  (In fact, we can improve the convergence result for topological solutions in Lemma 2.2).

**Lemma 2.1.** *Let  $(u_{\varepsilon,1}, u_{\varepsilon,2})$  be a sequence of solutions of (1.5). Then, up to subsequence, one of the following holds true:*

- (i) for  $i = 1, 2$ ,  $u_{\varepsilon,i} \rightarrow -\infty$  a.e. as  $\varepsilon \rightarrow 0$ ;
- (ii) for  $i = 1, 2$ ,  $u_{\varepsilon,i} \rightarrow 0$  a.e. as  $\varepsilon \rightarrow 0$ . Moreover,  $u_{\varepsilon,i} \rightarrow 0$  in  $L^p(\mathbb{T}^2)$  for some  $p > 1$ ,  $i = 1, 2$ .

*Proof.* By (2.1),  $e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) \rightarrow 0$  in  $L^1(\mathbb{T}^2)$  as  $\varepsilon \rightarrow 0$ . Hence, it is clear that either  $u_{\varepsilon,i} \rightarrow -\infty$  a.e. or  $u_{\varepsilon,i} \rightarrow 0$  a.e. for  $i = 1, 2$ . So, we only need to show the  $L^p$  estimate in (ii).

Let  $d_{\varepsilon,i} = \frac{1}{|\mathbb{T}^2|} \int_{\mathbb{T}^2} u_{\varepsilon,i} dx$  and  $u_{\varepsilon,i} = w_{\varepsilon,i} + u_{0,i} + d_{\varepsilon,i}$ . Then  $(w_{\varepsilon,1}, w_{\varepsilon,2})$  satisfies

$$\begin{cases} \Delta w_{\varepsilon,1} + \frac{1}{\varepsilon^2} e^{u_{\varepsilon,2}} (1 - e^{u_{\varepsilon,1}}) = \frac{4\pi N_1}{|\mathbb{T}^2|} & \text{on } \mathbb{T}^2, \\ \Delta w_{\varepsilon,2} + \frac{1}{\varepsilon^2} e^{u_{\varepsilon,1}} (1 - e^{u_{\varepsilon,2}}) = \frac{4\pi N_2}{|\mathbb{T}^2|} & \text{on } \mathbb{T}^2, \end{cases} \quad (2.2)$$

and  $\int_{\mathbb{T}^2} w_{\varepsilon,i} dx = 0$ ,  $i = 1, 2$ .

We claim that there exist  $C_q > 0$  such that  $\|\nabla w_{\varepsilon,i}\|_{L^q(\mathbb{T}^2)} \leq C_q$  for any  $q \in (1, 2)$ . Let  $q' = \frac{q}{q-1} > 2$ . Then

$$\begin{aligned} & \|\nabla w_{\varepsilon,i}\|_{L^q(\mathbb{T}^2)} \\ & \leq \sup \left\{ \left| \int_{\mathbb{T}^2} \nabla w_{\varepsilon,i} \nabla \phi dx \right| \mid \phi \in W^{1,q'}(\mathbb{T}^2), \int_{\mathbb{T}^2} \phi dx = 0, \|\phi\|_{W^{1,q'}(\mathbb{T}^2)} = 1 \right\}. \end{aligned} \quad (2.3)$$

By lemma 7.16 in [11], if  $\int_{\mathbb{T}^2} \phi dx = 0$ , then there exist  $c, C > 0$  such that

$$|\phi(x)| \leq c \int_{\mathbb{T}^2} \frac{|\nabla \phi|}{|x-y|} dy \leq C \|\nabla \phi\|_{L^{q'}(\mathbb{T}^2)} \quad \text{for } x \in \mathbb{T}^2. \quad (2.4)$$

Thus in view of (2.2), (2.4), and (2.1), we see that there exists constant  $C > 0$ , independent of  $\phi$  satisfying  $\int_{\mathbb{T}^2} \phi dx = 0$ ,  $\|\phi\|_{W^{1,q'}(\mathbb{T}^2)} = 1$ ,

$$\begin{aligned} \left| \int_{\mathbb{T}^2} \nabla w_{\varepsilon,i} \nabla \phi dx \right| &= \left| \int_{\mathbb{T}^2} \Delta w_{\varepsilon,i} \phi dx \right| \\ &\leq \|\phi\|_{L^\infty(\mathbb{T}^2)} \left| \int_{\mathbb{T}^2} \frac{1}{\varepsilon^2} e^{u_{\varepsilon,j}} (e^{u_{\varepsilon,i}} - 1) dx + 4\pi N_i \right| \leq C. \end{aligned} \quad (2.5)$$

Now using (2.3), we complete the proof of our claim.

In view of Poincaré inequality, we also have  $\|w_{\varepsilon,i}\|_{L^q(\mathbb{T}^2)} \leq c \|\nabla w_{\varepsilon,i}\|_{L^q(\mathbb{T}^2)}$ . Then there exist  $w_i \in W^{1,q}(\mathbb{T}^2)$  and  $p > 1$  such that, as  $\varepsilon \rightarrow 0$ ,

$$w_{\varepsilon,i} \rightharpoonup w_i \text{ weakly in } W^{1,q}(\mathbb{T}^2), \quad w_{\varepsilon,i} \rightarrow w_i \text{ strongly in } L^p(\mathbb{T}^2), \quad w_{\varepsilon,i} \rightarrow w_i \text{ a.e.} \quad (2.6)$$

We consider the following possible cases.

- (i)  $\limsup_{\varepsilon \rightarrow 0} \frac{e^{d_{\varepsilon,i}}}{\varepsilon^2} \leq c$  for some constant  $c > 0$ .
- (ii)  $\limsup_{\varepsilon \rightarrow 0} \frac{e^{d_{\varepsilon,i}}}{\varepsilon^2} = +\infty$ .

If  $\limsup_{\varepsilon \rightarrow 0} \frac{e^{d_{\varepsilon,i}}}{\varepsilon^2}$  is bounded, then

$$e^{u_{\varepsilon,i}} = e^{w_{\varepsilon,i} + d_{\varepsilon,i} + u_{0,i}} \leq c\varepsilon^2 e^{w_{\varepsilon,i} + u_{0,i}} \rightarrow 0 \quad \text{a.e. as } \varepsilon \rightarrow 0,$$

which implies that  $u_{\varepsilon,i} \rightarrow -\infty$  a.e. as  $\varepsilon \rightarrow 0$ .

Next, we consider the case

$$\limsup_{\varepsilon \rightarrow 0} \frac{e^{d_{\varepsilon,i}}}{\varepsilon^2} = +\infty. \quad (2.7)$$

Since  $u_{\varepsilon,i} < 0$  on  $\mathbb{T}^2$ , we see that  $0 \leq e^{d_{\varepsilon,i}} \leq 1$  which implies there exists  $A_i \geq 0$  such that  $\limsup_{\varepsilon \rightarrow 0} e^{d_{\varepsilon,i}} = A_i$ . By using Fatou's lemma and (2.6), we see that

$$4\pi N_i \varepsilon^2 = \int_{\mathbb{T}^2} e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) dx \geq \int_{\mathbb{T}^2} A_j e^{w_j + u_{0,j}} (1 - A_i e^{w_i + u_{0,i}}) dx,$$

which implies that  $A_j \equiv 0$  or  $w_i + u_{0,i} = -\ln A_i$  a.e. in  $\mathbb{T}^2$ .



Let  $1 \leq i \neq j \leq 2$ . If  $A_j \equiv 0$ , then  $A_i \equiv 0$  by the same argument. Thus,  $\lim_{\varepsilon \rightarrow 0} d_{\varepsilon,i} = -\infty$ . Moreover, by using (2.6), we get that  $u_{\varepsilon,i} = w_{\varepsilon,i} + d_{\varepsilon,i} + u_{0,i} \rightarrow -\infty$  a.e. in  $\mathbb{T}^2$  for  $i = 1, 2$ .

If  $w_i + u_{0,i} = -\ln A_i$  a.e. in  $\mathbb{T}^2$ , then in view of  $\int_{\mathbb{T}^2} w_{\varepsilon,i} + u_{0,i} dx = 0$ , we see that  $A_i = 1$  and  $w_i + u_{0,i} = 0$  a.e. in  $\mathbb{T}^2$ . Thus,  $u_{\varepsilon,i} = w_{\varepsilon,i} + d_{\varepsilon,i} + u_{0,i} \rightarrow w_i + \ln A_i + u_{0,i} = 0$  a.e. in  $\mathbb{T}^2$  and  $u_{\varepsilon,i} \rightarrow 0$  in  $L^p(\mathbb{T}^2)$  for some  $p > 1$ . Now we complete the proof of Lemma 2.1.  $\square$

In the following lemma, we get the detailed information about topological solutions.

**Lemma 2.2.** *Let  $(u_{\varepsilon,1}, u_{\varepsilon,2})$  be a sequence of topological solutions of (1.5) Then we have as  $\varepsilon \rightarrow 0$ ,*

- (i)  $u_{\varepsilon,i} \rightarrow 0$  in  $C_{loc}^m(\mathbb{T}^2 \setminus Z)$  for any  $m \in \mathbb{Z}^+$  and faster than any other power of  $\varepsilon > 0$ ;
- (ii) for  $1 \leq j \neq i \leq 2$ ,  $\frac{1}{\varepsilon^2} e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) \rightarrow 4\pi \sum_{j=1}^{N_i} \delta_{p_{j,i}}$  and  $\frac{1}{\varepsilon^2} (1 - e^{u_{\varepsilon,1}})(1 - e^{u_{\varepsilon,2}}) \rightarrow 4\pi \left( \sum_{j=1}^{N_1} \delta_{p_{j,1}} \right) \left( \sum_{j=1}^{N_2} \delta_{p_{j,2}} \right)$  weakly in the sense of measure in  $\mathbb{T}^2$ .

*Proof.* Denote  $\mathbb{T}_\delta^2 \equiv \{x \in \mathbb{T}^2 \mid \text{dist}(x, p_{j,i}) \geq \delta \text{ for all } i, j\}$ . Since  $u_{\varepsilon,i} < 0$  on  $\mathbb{T}^2$ , we note that  $u_{\varepsilon,i}$  is subharmonic in  $\mathbb{T}_\delta^2$ . By using the mean value theorem and Lemma 2.1, we have that

$$0 \leq -u_{\varepsilon,i} \leq \frac{1}{|\mathbb{T}_\delta^2|} \|u_{\varepsilon,i}\|_{L^1(\mathbb{T}_\delta^2)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \text{ on } \mathbb{T}_{2\delta}^2. \quad (2.8)$$

We also have the following inequality,

$$\frac{|t|}{1+|t|} \leq |1 - e^t| \text{ for any } t \in \mathbb{R}. \quad (2.9)$$

By using (2.1) and (2.8), we deduce the estimate

$$\begin{aligned} \int_{\mathbb{T}_\delta^2} |u_{\varepsilon,i}| dx &\leq (1 + \|u_{\varepsilon,i}\|_{L^\infty(\mathbb{T}_\delta^2)}) \int_{\mathbb{T}_\delta^2} \frac{|u_{\varepsilon,i}|}{1 + |u_{\varepsilon,i}|} dx \\ &\leq (1 + \|u_{\varepsilon,i}\|_{L^\infty(\mathbb{T}_\delta^2)}) e^{\|u_{\varepsilon,i}\|_{L^\infty(\mathbb{T}_\delta^2)}} \int_{\mathbb{T}_\delta^2} e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) dx \\ &\leq 8\pi e N_i \varepsilon^2. \end{aligned} \quad (2.10)$$

Let  $\phi \in C^\infty(\overline{\mathbb{T}^2})$  satisfy

$$\begin{cases} \phi = 0 & \text{in } \mathbb{T}^2 \setminus \mathbb{T}_\delta^2 \\ \phi = 1 & \text{in } \mathbb{T}_{2\delta}^2 \end{cases} \quad (2.11)$$

and  $0 \leq \phi \leq 1$ . By using  $\phi$  as a test function in (1.5) and (2.10), we get that

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{\mathbb{T}_{2\delta}^2} e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) dx &\leq \frac{1}{\varepsilon^2} \int_{\mathbb{T}^2} e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) \phi dx \\ &= - \int_{\mathbb{T}^2} \Delta u_{\varepsilon,i} \phi dx = - \int_{\mathbb{T}^2} u_{\varepsilon,i} \Delta \phi dx \\ &\leq c_\delta \|u_{\varepsilon,i}\|_{L^1(\mathbb{T}_\delta^2)} \leq C_\delta \varepsilon^2, \end{aligned} \quad (2.12)$$

for some constants  $c_\delta, C_\delta > 0$ . By a suitable iteration of (2.10), (2.12), and the elliptic estimates, we deduce that (i) holds. In other words, for any small  $\delta > 0$  and any  $m, n \in \mathbb{Z}^+$ , there exists a constant  $c_{\delta,m,n} > 0$  such that

$$\sup_{\mathbb{T}_{2\delta}^2} \left( \sum_{|\alpha|=0}^m |D^\alpha u_{\varepsilon,i}| \right) \leq c_{\delta,m,n} \varepsilon^n. \quad (2.13)$$

Next, if we take  $\phi \in C^\infty(\mathbb{T}^2)$  as a test function into (1.5), from Lemma 2.1, we see that

$$\begin{aligned} &\left| \int_{\mathbb{T}^2} \frac{1}{\varepsilon^2} e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) \phi dx - 4\pi \sum_{j=1}^{N_i} \phi(p_{j,i}) \right| \\ &= \left| \int_{\mathbb{T}^2} -\Delta u_{\varepsilon,i} \phi dx \right| = \left| \int_{\mathbb{T}^2} -u_{\varepsilon,i} \Delta \phi dx \right| \\ &\leq \|\phi\|_{C^2(\mathbb{T}^2)} \|u_{\varepsilon,i}\|_{L^1(\mathbb{T}^2)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \quad (2.14)$$

Choose small  $r > 0$  such that  $B_r(p_{j,i}) \cap B_r(p_{j',i'}) = \emptyset$  if  $p_{j,i} \neq p_{j',i'}$  and let

$$v_{\varepsilon,i}(x) = u_{\varepsilon,i}(x) - 2\nu_i \ln |x - p| \quad \text{on } B_r(p),$$

where  $\nu_i = 0$  if  $p \notin \cup_{j=1}^{d_i} \{p_{j,i}\}$  and  $\nu_i = \#\{p_{j,i} | p_{j,i} = p\}$ . Then  $v_{\varepsilon,i}$  satisfies

$$\Delta v_{\varepsilon,i} + \frac{1}{\varepsilon^2} e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) = 0 \quad \text{on } B_r(p). \quad (2.15)$$

For the sake of simplicity, we assume that  $p = 0$ . Multiplying (2.15) by  $\nabla u_{\varepsilon,j} \cdot x$  ( $1 \leq j \neq i \leq 2$ ) and integrating over  $B_r(0)$  (see [5]), we obtain the Pohozaev type identity

$$\begin{aligned} &\int_{\partial B_r(0)} \left[ \frac{2(\nabla u_{\varepsilon,1} \cdot x)(\nabla u_{\varepsilon,2} \cdot x)}{|x|} - (\nabla u_{\varepsilon,1} \cdot \nabla u_{\varepsilon,2})|x| - \frac{1}{\varepsilon^2} (1 - e^{u_{\varepsilon,1}})(1 - e^{u_{\varepsilon,2}})|x| \right] d\sigma \\ &= - \int_{B_r(0)} \frac{2}{\varepsilon^2} (1 - e^{u_{\varepsilon,1}})(1 - e^{u_{\varepsilon,2}}) dx + 8\pi \nu_1 \nu_2. \end{aligned}$$

By using (2.13), we have

$$\lim_{\varepsilon \rightarrow 0} \int_{B_r(p)} \frac{2}{\varepsilon^2} (1 - e^{u_{\varepsilon,1}})(1 - e^{u_{\varepsilon,2}}) dx = 8\pi \nu_1 \nu_2.$$

Now we complete the proof of Lemma 2.2. □

The existence of topological solution (in fact, maximal solution) of (1.5) can be proved by Lemma 2.1 and [20, Theorem 1.1-(i),(ii)]. Hence, to prove Theorem 1.1, it suffices to prove the uniqueness property. To prove Theorem 1.1, we argue by contradiction and suppose that there exist two sequences of distinct topological solutions  $(u_{\varepsilon,1}, u_{\varepsilon,2})$  and  $(\tilde{u}_{\varepsilon,1}, \tilde{u}_{\varepsilon,2})$  of (1.5). Without loss of generality, we may assume that there exists  $x_\varepsilon \in \mathbb{T}^2$  such that

$$|u_{\varepsilon,1}(x_\varepsilon) - \tilde{u}_{\varepsilon,1}(x_\varepsilon)| = \|u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)} \geq \|u_{\varepsilon,2} - \tilde{u}_{\varepsilon,2}\|_{L^\infty(\mathbb{T}^2)},$$

and  $x_\varepsilon \rightarrow p$  for some  $p$  in  $\mathbb{T}^2$ . Set  $A_\varepsilon \equiv \frac{u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}}{\|u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)}}$  and  $B_\varepsilon \equiv \frac{u_{\varepsilon,2} - \tilde{u}_{\varepsilon,2}}{\|u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)}}$ . Then  $(A_\varepsilon, B_\varepsilon)$  satisfies

$$\begin{cases} \Delta A_\varepsilon - \frac{1}{\varepsilon^2} e^{\tilde{u}_{\varepsilon,2} + \eta_{\varepsilon,1}} A_\varepsilon + \frac{1}{\varepsilon^2} e^{\eta_{\varepsilon,2}} (1 - e^{u_{\varepsilon,1}}) B_\varepsilon = 0 & \text{on } \mathbb{T}^2, \\ \Delta B_\varepsilon - \frac{1}{\varepsilon^2} e^{\tilde{u}_{\varepsilon,1} + \eta_{\varepsilon,2}} B_\varepsilon + \frac{1}{\varepsilon^2} e^{\eta_{\varepsilon,1}} (1 - e^{u_{\varepsilon,2}}) A_\varepsilon = 0 & \text{on } \mathbb{T}^2, \end{cases} \quad (2.16)$$

where  $\eta_{\varepsilon,i}$  is between  $u_{\varepsilon,i}$  and  $\tilde{u}_{\varepsilon,i}$ ,  $i = 1, 2$ . Choose small  $r_0 > 0$  such that  $B_{r_0}(p_{j,i}) \cap B_{r_0}(p_{j',i'}) = \emptyset$  if  $p_{j,i} \neq p_{j',i'}$ . We consider the scaled functions

$$\hat{u}_{\varepsilon,i}(y) = u_{\varepsilon,i}(\varepsilon y + p), \quad \bar{u}_{\varepsilon,i}(y) = \tilde{u}_{\varepsilon,i}(\varepsilon y + p) \quad \text{in } B_{\frac{r_0}{\varepsilon}}(0) \equiv \left\{ y \in \mathbb{R}^2 \mid |y| < \frac{r_0}{\varepsilon} \right\}.$$

Then both  $(\hat{u}_{\varepsilon,1}, \hat{u}_{\varepsilon,2})$  and  $(\bar{u}_{\varepsilon,1}, \bar{u}_{\varepsilon,2})$  are solutions of

$$\begin{cases} \Delta u_{\varepsilon,1} + e^{u_{\varepsilon,2}} (1 - e^{u_{\varepsilon,1}}) = 4\pi\nu_1\delta_0 & \text{on } B_{\frac{r_0}{\varepsilon}}(0), \\ \Delta u_{\varepsilon,2} + e^{u_{\varepsilon,1}} (1 - e^{u_{\varepsilon,2}}) = 4\pi\nu_2\delta_0 & \text{on } B_{\frac{r_0}{\varepsilon}}(0), \end{cases}$$

where  $\nu_i = 0$  if  $p \notin \cup_{j=1}^{d_i} \{p_{j,i}\}$  and  $\nu_i = \#\{p_{j,i} \mid p_{j,i} = p\}$ .

We show the gradient estimate for the topological solutions to (1.5) in the following lemma.

**Lemma 2.3.** *There exists a constant  $c > 0$ , independent of  $r > 0$  and  $\varepsilon > 0$ , such that*

$$\left| \nabla \hat{u}_{\varepsilon,i}(x) - \frac{2\nu_i x}{|x|^2} \right| + \left| \nabla \bar{u}_{\varepsilon,i}(x) - \frac{2\nu_i x}{|x|^2} \right| \leq c \quad \text{on } B_{\frac{r}{\varepsilon}}(0) \text{ for } i = 1, 2.$$

*Proof.* We remind the Green's function  $G$  on  $\mathbb{T}$  which satisfies

$$-\Delta_x G(x, y) = \delta_y - \frac{1}{|\mathbb{T}|}, \quad x, y \in \mathbb{T} \text{ and } \int_{\mathbb{T}} G(x, y) dx = 0. \quad (2.17)$$

And we denote by  $\gamma(x, y) = G(x, y) + \frac{1}{2\pi} \ln |x - y|$  the regular part of  $G$ . We also recall that

$$u_{0,i} = -4\pi \sum_{j=1}^{N_i} G(x, p_{j,i}), \quad i = 1, 2. \quad (2.18)$$

Then by using the Green's representation formula for a solution  $(u_{\varepsilon,1}, u_{\varepsilon,2})$  of (1.5), we see that for  $x \in \mathbb{T}$ ,

$$u_{\varepsilon,i}(x) - u_{0,i}(x) = \frac{1}{|\mathbb{T}|} \int_{\mathbb{T}} u_{\varepsilon,i}(y) dy + \int_{\mathbb{T}} G(x, y) \frac{1}{\varepsilon^2} e^{u_{\varepsilon,j}} (1 - e^{u_{\varepsilon,i}}) dy. \quad (2.19)$$

Then we see that for  $x \in B_r(p)$ ,

$$\begin{aligned} & \left| \nabla u_{\varepsilon,i}(x) - \frac{2\nu_i(x-p)}{|x-p|^2} \right| \\ & \leq C + \frac{1}{2\pi\varepsilon^2} \int_{\mathbb{T}^2} \frac{e^{\hat{u}_{\varepsilon,j}} (1 - e^{\hat{u}_{\varepsilon,i}})}{|x-y|} dy \\ & = C + \frac{1}{2\pi\varepsilon^2} \left( \int_{B_\varepsilon(x)} \frac{e^{\hat{u}_{\varepsilon,j}} (1 - e^{\hat{u}_{\varepsilon,i}})}{|x-y|} dy + \int_{\mathbb{T}^2 \setminus B_\varepsilon(x)} \frac{e^{\hat{u}_{\varepsilon,j}} (1 - e^{\hat{u}_{\varepsilon,i}})}{|x-y|} dy \right) \\ & \leq C + \frac{C'}{\varepsilon}, \end{aligned}$$

for some constants  $C, C' > 0$ , independent of  $r > 0$  and  $\varepsilon > 0$ . The desired conclusion follows by the substitution  $x = \varepsilon x + p$ ,  $\hat{u}_{\varepsilon,i}(x) = u_{\varepsilon,i}(\varepsilon x + p)$  and  $\bar{u}_{\varepsilon,i}(x) = \tilde{u}_{\varepsilon,i}(\varepsilon x + p)$ .  $\square$

**Lemma 2.4.**  $\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^2 \left( \sup_{B_{\frac{r_0}{\varepsilon}}(0)} (|\hat{u}_{\varepsilon,i} - u_i| + |\bar{u}_{\varepsilon,i} - u_i|) \right) = 0$ , where  $(u_1, u_2)$  is a unique topological solution of

$$\begin{cases} \Delta u_i + e^{u_j} (1 - e^{u_i}) = 4\pi\nu_i \delta_0 \text{ in } \mathbb{R}^2, \quad 1 \leq j \neq i \leq 2; \\ u_i < 0, \quad \sup_{\mathbb{R}^2 \setminus B_1(0)} |\nabla u_i| < +\infty \quad i = 1, 2; \\ e^{u_2} (1 - e^{u_1}), \quad e^{u_1} (1 - e^{u_2}), \quad (1 - e^{u_1})(1 - e^{u_2}) \in L^1(\mathbb{R}^2). \end{cases} \quad (2.20)$$

*Proof.* We decompose

$$\hat{u}_{\varepsilon,i}(y) = 2\nu_i \ln |y| + \hat{v}_{\varepsilon,i}(y). \quad (2.21)$$

Then  $\hat{v}_{\varepsilon,i}$  ( $i = 1, 2$ ) satisfies

$$\Delta \hat{v}_{\varepsilon,i} + |y|^{2\nu_j} e^{\hat{v}_{\varepsilon,j}} (1 - |y|^{2\nu_i} e^{\hat{v}_{\varepsilon,i}}) = 0 \text{ in } B_{\frac{r_0}{\varepsilon}}(0), \quad (2.22)$$

where  $1 \leq j \neq i \leq 2$ . Since  $\hat{u}_{\varepsilon,i} = 2\nu_i \ln |y| + \hat{v}_{\varepsilon,i} < 0$  on  $B_{\frac{r_0}{\varepsilon}}(0)$ , we have

$$\hat{v}_{\varepsilon,i} \Big|_{\partial B_{\frac{r_0}{\varepsilon}}(0)} < -2\nu_i \ln R \quad \text{for any } R > 0.$$

By using the Green's representation formula for a solution  $u_{\varepsilon,i}$  of (1.5) (see Lemma 2.3), we see that there exists  $c_0 > 0$  such that

$$|\nabla \hat{v}_{\varepsilon,i}(x)| \leq c_0 \quad \text{on } B_{\frac{r_0}{\varepsilon}}(0). \quad (2.23)$$

We claim that  $\hat{v}_{\varepsilon,i}$  is uniformly bounded in the  $C^{2,\alpha}$  topology. To prove our claim, we argue by contradiction and suppose that there exists  $R_0 > 0$  such that

$$\lim_{\varepsilon \rightarrow 0} \left( \inf_{B_{R_0}(0)} \hat{v}_{\varepsilon,i} \right) = -\infty.$$

Then (2.23) implies that  $\lim_{\varepsilon \rightarrow 0} \left( \sup_{B_R(0)} \hat{v}_{\varepsilon,i} \right) = -\infty$  for any  $R \geq R_0$ . Clearly Lemma 2.2 shows that, for any  $R \geq R_0$ ,

$$\begin{aligned} 8\pi(\nu_1\nu_2 + \nu_i) &= \lim_{\varepsilon \rightarrow 0} \int_{B_{\frac{r_0}{\varepsilon}}(0)} \frac{2}{\varepsilon^2} (1 - e^{u_{\varepsilon,i}}) dx \geq \lim_{\varepsilon \rightarrow 0} \int_{B_R(0)} 2(1 - e^{\hat{v}_{\varepsilon,i}}) dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_{B_R(0)} 2(1 - |x|^{2\nu_i} e^{\hat{v}_{\varepsilon,i}}) dx \geq \pi R^2. \end{aligned} \quad (2.24)$$

Since the right hand side of (2.24) could be arbitrarily large, we obtain a contradiction which proves our claim.

Then we obtain a subsequence  $\hat{v}_{\varepsilon,i}$  (still denoted in the same way) such that

$$\hat{v}_{\varepsilon,i} \rightarrow v_i \quad \text{uniformly in } C_{\text{loc}}^2(\mathbb{R}^2). \quad (2.25)$$

Let us define  $u_i(y) \equiv 2\nu_i \ln |y| + v_i(y)$ . In view of (2.23), (2.1) and Lemma 2.2, we see that  $(u_1, u_2)$  satisfies (2.20). Since  $\sup_{\mathbb{R}^2 \setminus B_1(0)} |\nabla u_i| < +\infty$  and  $1 - e^{u_i} \in L^1(\mathbb{R}^2)$ , we see that  $(u_1, u_2)$  is a topological solution in  $\mathbb{R}^2$ . Indeed, if there exists a sequence  $x_n \in \mathbb{R}^2$  such that,

$$\lim_{n \rightarrow \infty} |x_n| \rightarrow +\infty, \quad \lim_{n \rightarrow \infty} u_i(x_n) = c \neq 0,$$

then since  $\sup_{|x| \geq 1} |\nabla u_i(x)| \leq C$ , there exist small  $r_1 > 0$  and  $c_0 > 0$ , independent of  $n$ , such that

$$1 - e^{u_i} \geq c_0 > 0 \quad \text{on } B_{r_1}(x_n).$$

Then  $\int_{\mathbb{R}^2} (1 - e^{u_i}) dx \geq \sum_{n=1}^{\infty} \int_{B_{r_1}(x_n)} (1 - e^{u_i}) dx = +\infty$  which is a contradiction. Thus,  $(u_1, u_2)$  is a topological solution of (2.20) in  $\mathbb{R}^2$ .

Moreover, by using a Pohozaev type identity (see Lemma 2.2), we have

$$\int_{\mathbb{R}^2} (1 - e^{u_1})(1 - e^{u_2}) dx = 4\pi\nu_1\nu_2 \quad \text{and} \quad \int_{\mathbb{R}^2} (1 - e^{u_i}) dx = 4\pi(\nu_1\nu_2 + \nu_i). \quad (2.26)$$

By [19], we also see that  $u_i$  admits exponential decay at infinity. Then in view of Lemma 2.2, (2.26), and the dominated convergence theorem, we get that

$$0 = \lim_{\varepsilon \rightarrow 0} \int_{B_{\frac{r_0}{\varepsilon}}(0)} |e^{\hat{u}_{\varepsilon,i}} - e^{u_i}| dx \geq \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \int_{B_{\frac{r_0}{\varepsilon}}(0)} |\hat{u}_{\varepsilon,i} - u_i| dx,$$

which implies that

$$\lim_{\varepsilon \rightarrow 0} \left( \sup_{B_{\frac{r_0}{\varepsilon}}(0)} |\hat{u}_{\varepsilon,i} - u_i| \right) = 0$$

from (2.25),  $\sup_{\mathbb{R}^2 \setminus B_1(0)} |\nabla u_i| < +\infty$ , and (2.23).

By Theorem B, we know that a topological solution of (2.20) is unique. So, by applying the above arguments to  $(\bar{u}_{\varepsilon,1}, \bar{u}_{\varepsilon,2})$ , we complete the proof of Lemma 2.4.  $\square$

### 3. PROOF OF THEOREM 1.1-1.2

Firstly, we focus on the proof of Theorem 1.1.

**Proof of Theorem 1.1** Recall that

$$A_\varepsilon \equiv \frac{u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}}{\|u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)}} \quad \text{and} \quad B_\varepsilon \equiv \frac{u_{\varepsilon,2} - \tilde{u}_{\varepsilon,2}}{\|u_{\varepsilon,1} - \tilde{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)}},$$

and  $(A_\varepsilon, B_\varepsilon)$  satisfies (1.13). We consider the following two possible cases.

*Case 1.*  $\lim_{\varepsilon \rightarrow 0} \frac{|x_\varepsilon - p|}{\varepsilon} < +\infty$ :

In this case, there exists  $x_0 \in \mathbb{R}^2$  such that  $\lim_{\varepsilon \rightarrow 0} \frac{x_\varepsilon - p}{\varepsilon} = x_0$ . Let  $\hat{A}_\varepsilon(y) \equiv A_\varepsilon(\varepsilon y + p)$  and  $\hat{B}_\varepsilon(y) \equiv B_\varepsilon(\varepsilon y + p)$ . Then  $(\hat{A}_\varepsilon, \hat{B}_\varepsilon)$  satisfies

$$\begin{cases} \Delta \hat{A}_\varepsilon - e^{\bar{u}_{\varepsilon,2} + \hat{\eta}_{\varepsilon,1}} \hat{A}_\varepsilon + e^{\hat{\eta}_{\varepsilon,2}} (1 - e^{\hat{u}_{\varepsilon,1}}) \hat{B}_\varepsilon = 0 & \text{on } B_{\frac{r_0}{\varepsilon}}(0), \\ \Delta \hat{B}_\varepsilon - e^{\bar{u}_{\varepsilon,1} + \hat{\eta}_{\varepsilon,2}} \hat{B}_\varepsilon + e^{\hat{\eta}_{\varepsilon,1}} (1 - e^{\hat{u}_{\varepsilon,2}}) \hat{A}_\varepsilon = 0 & \text{on } B_{\frac{r_0}{\varepsilon}}(0), \end{cases}$$

where  $\hat{\eta}_{\varepsilon,i}$  is between  $\hat{u}_{\varepsilon,i}$  and  $\bar{u}_{\varepsilon,i}$ ,  $i = 1, 2$ . Then we obtain a subsequence  $(\hat{A}_\varepsilon, \hat{B}_\varepsilon)$  (still denoted in the same way) such that

$$(\hat{A}_\varepsilon, \hat{B}_\varepsilon) \rightarrow (\hat{A}, \hat{B}) \text{ uniformly in } C_{\text{loc}}^2(\mathbb{R}^2) \times C_{\text{loc}}^2(\mathbb{R}^2),$$

where  $(\hat{A}, \hat{B})$  is a bounded solution of

$$\begin{cases} \Delta \hat{A} - e^{u_1+u_2} \hat{A} + e^{u_2}(1 - e^{u_1}) \hat{B} = 0 & \text{on } \mathbb{R}^2, \\ \Delta \hat{B} - e^{u_1+u_2} \hat{B} + e^{u_1}(1 - e^{u_2}) \hat{A} = 0 & \text{on } \mathbb{R}^2. \end{cases}$$

By Theorem B, we obtain  $(\hat{A}, \hat{B}) \equiv (0, 0)$ . However, we see that

$$\begin{aligned} (0, 0) &= (\hat{A}(x_0), \hat{B}(x_0)) \\ &= \lim_{\varepsilon \rightarrow 0} \left( \hat{A}_\varepsilon \left( \frac{x_\varepsilon - p}{\varepsilon} \right), \hat{B}_\varepsilon \left( \frac{x_\varepsilon - p}{\varepsilon} \right) \right) \\ &= \lim_{\varepsilon \rightarrow 0} (A_\varepsilon(x_\varepsilon), B_\varepsilon(x_\varepsilon)), \end{aligned}$$

where  $|A_\varepsilon(x_\varepsilon)| = \frac{|u_{\varepsilon,1}(x_\varepsilon) - \bar{u}_{\varepsilon,1}(x_\varepsilon)|}{\|u_{\varepsilon,1} - \bar{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)}} = 1$  from the choice of  $x_\varepsilon$ . It is a contradiction.

*Case 2.*  $\lim_{\varepsilon \rightarrow 0} \frac{|x_\varepsilon - p|}{\varepsilon} = +\infty$ :

Let

$$\begin{cases} \hat{u}_{\varepsilon,i}(y) = u_{\varepsilon,i}(\varepsilon y + x_\varepsilon) - 2\nu_i \ln |\varepsilon y + x_\varepsilon - p| + 2\nu_i \ln |x_\varepsilon - p| & \text{on } B_{\frac{|x_\varepsilon - p|}{2\varepsilon}}(0), \\ \bar{\bar{u}}_{\varepsilon,i}(y) = \bar{u}_{\varepsilon,i}(\varepsilon y + x_\varepsilon) - 2\nu_i \ln |\varepsilon y + x_\varepsilon - p| + 2\nu_i \ln |x_\varepsilon - p| & \text{on } B_{\frac{|x_\varepsilon - p|}{2\varepsilon}}(0), \end{cases}$$

where  $\nu_i = 0$  if  $p \notin \cup_{j=1}^{d_i} \{p_{j,i}\}$  and  $\nu_i = \#\{p_{j,i} | p_{j,i} = p\}$ . Then both  $(\hat{u}_{\varepsilon,1}, \hat{u}_{\varepsilon,2})$  and  $(\bar{\bar{u}}_{\varepsilon,1}, \bar{\bar{u}}_{\varepsilon,2})$  are solutions of

$$\begin{cases} \Delta u_{\varepsilon,1} + \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2\nu_2} e^{u_{\varepsilon,2}} \left( 1 - \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2\nu_1} e^{u_{\varepsilon,1}} \right) = 0 & \text{on } B_{\frac{|x_\varepsilon - p|}{2\varepsilon}}(0), \\ \Delta u_{\varepsilon,2} + \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2\nu_1} e^{u_{\varepsilon,1}} \left( 1 - \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2\nu_2} e^{u_{\varepsilon,2}} \right) = 0 & \text{on } B_{\frac{|x_\varepsilon - p|}{2\varepsilon}}(0). \end{cases}$$

Then the previous arguments, we see that  $\lim_{\varepsilon \rightarrow 0} \sum_{i=1}^2 \left( \sup_{B_{\frac{r_0}{\varepsilon}}(0)} (|\hat{u}_{\varepsilon,i} - u_i| + |\bar{\bar{u}}_{\varepsilon,i} - u_i|) \right) = 0$ , where  $(u_1, u_2)$  is a unique topological solution of

$$\Delta u_i + e^{u_j}(1 - e^{u_i}) = 0 \text{ in } \mathbb{R}^2, \quad 1 \leq j \neq i \leq 2.$$

Let  $\hat{A}_\varepsilon(y) \equiv A_\varepsilon(\varepsilon y + x_\varepsilon)$  and  $\hat{B}_\varepsilon(y) \equiv B_\varepsilon(\varepsilon y + x_\varepsilon)$ . Then on  $B_{\frac{|x_\varepsilon - p|}{2\varepsilon}}(0)$ ,  $(\hat{A}_\varepsilon, \hat{B}_\varepsilon)$  satisfies

$$\begin{cases} \Delta \hat{A}_\varepsilon - \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2(\nu_1 + \nu_2)} e^{\bar{u}_{\varepsilon,2} + \hat{\eta}_{\varepsilon,1}} \hat{A}_\varepsilon + \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2\nu_2} e^{\hat{\eta}_{\varepsilon,2}} (1 - \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2\nu_1} e^{\hat{u}_{\varepsilon,1}}) \hat{B}_\varepsilon = 0, \\ \Delta \hat{B}_\varepsilon - \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2(\nu_1 + \nu_2)} e^{\bar{u}_{\varepsilon,1} + \hat{\eta}_{\varepsilon,2}} \hat{B}_\varepsilon + \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2\nu_1} e^{\hat{\eta}_{\varepsilon,1}} (1 - \left| \frac{\varepsilon y + x_\varepsilon - p}{|x_\varepsilon - p|} \right|^{2\nu_2} e^{\hat{u}_{\varepsilon,2}}) \hat{A}_\varepsilon = 0, \end{cases}$$

where  $\hat{\eta}_{\varepsilon,i}$  is between  $\hat{u}_{\varepsilon,i}$  and  $\bar{u}_{\varepsilon,i}$ . Then we obtain a subsequence  $(\hat{A}_\varepsilon, \hat{B}_\varepsilon)$  (still denoted in the same way) such that

$$(\hat{A}_\varepsilon, \hat{B}_\varepsilon) \rightarrow (\hat{A}, \hat{B}) \text{ uniformly in } C_{\text{loc}}^2(\mathbb{R}^2) \times C_{\text{loc}}^2(\mathbb{R}^2),$$

where  $(\hat{A}, \hat{B})$  is a bounded solution of

$$\begin{cases} \Delta \hat{A} - e^{u_1 + u_2} \hat{A} + e^{u_2} (1 - e^{u_1}) \hat{B} = 0 & \text{on } \mathbb{R}^2, \\ \Delta \hat{B} - e^{u_1 + u_2} \hat{B} + e^{u_1} (1 - e^{u_2}) \hat{A} = 0 & \text{on } \mathbb{R}^2. \end{cases}$$

Then Theorem B implies  $(\hat{A}, \hat{B}) = (0, 0)$ . However, we see that

$$\begin{aligned} (0, 0) &= (\hat{A}(0), \hat{B}(0)) \\ &= \lim_{\varepsilon \rightarrow 0} (\hat{A}_\varepsilon(0), \hat{B}_\varepsilon(0)) \\ &= \lim_{\varepsilon \rightarrow 0} (A_\varepsilon(x_\varepsilon), B_\varepsilon(x_\varepsilon)), \end{aligned}$$

where  $|A_\varepsilon(x_\varepsilon)| = \frac{|u_{\varepsilon,1}(x_\varepsilon) - \bar{u}_{\varepsilon,1}(x_\varepsilon)|}{\|u_{\varepsilon,1} - \bar{u}_{\varepsilon,1}\|_{L^\infty(\mathbb{T}^2)}} = 1$  from the choice of  $x_\varepsilon$ , and we get a contradiction.

So Theorem 1.1 is proved.  $\square$

**Proof of Theorem 1.2** For the proof of Theorem 1.2, the main part of difference from the proof of Theorem 1.1 is that we need to consider the case: maximum point  $x_\varepsilon$  of  $|u_{\varepsilon,i} - \bar{u}_{\varepsilon,i}|$  diverge to  $\infty$ . In this case, in view of Theorem A, we have

$$u_{\varepsilon,i}(\varepsilon x + x_\varepsilon) \rightarrow 0 \quad \text{in } C_{\text{loc}}^2(B_{\frac{r}{\varepsilon}}(0)).$$

Moreover, by using Lemma 2.4, we see that

$$(\tilde{A}_\varepsilon(y), \tilde{B}_\varepsilon) := (A_\varepsilon(\varepsilon y + x_\varepsilon), B_\varepsilon(\varepsilon y + x_\varepsilon)) \rightarrow (\tilde{A}, \tilde{B}) \text{ uniformly in } C_{\text{loc}}^2(\mathbb{R}^2) \times C_{\text{loc}}^2(\mathbb{R}^2),$$



where  $(\tilde{A}, \tilde{B})$  is a bounded solution of

$$\begin{cases} \Delta \tilde{A} - \tilde{A} = 0 & \text{on } \mathbb{R}^2, \\ \Delta \tilde{B} - \tilde{B} = 0 & \text{on } \mathbb{R}^2. \end{cases}$$

Then Theorem B implies  $(\tilde{A}, \tilde{B}) = (0, 0)$  which contradicts  $1 = \lim_{\varepsilon \rightarrow 0} |A_\varepsilon(x_\varepsilon)| = |\tilde{A}(0)|$ . Now we also complete the proof of Theorem 1.2.  $\square$

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